LINEAR ALGEBRA [M. MATH I YEAR]

MIDTERM EXAM

TOTAL MARKS : 30

Date: September 09, 2025

This exam is of total 30 marks and has a duration of 2 hours (11 AM – 1 PM). Please **read all questions carefully**. You may use any theorems learned in class, provided you clearly state them before applying.

1. Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and null space of T are disjoint, that is, have only the zero vector in common. [2 marks]

2. Let p, m, n be positive integers and $\mathbb{F}$ a field. Let V be the space of $m \times n$ matrices over $\mathbb{F}$ and W the space of $p \times n$ matrices over $\mathbb{F}$. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by
$$T(A) = BA.$$
Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix. [3 marks]

OR

If N is a nilpotent linear operator on V, show that for any polynomial f the semi-simple part of $f(N)$ is a scalar multiple of the identity operator (F a subfield of $\mathbb{C}$). [3 marks]

3. Let T be a linear operator on a finite-dimensional space over a subfield of $\mathbb{C}$. Prove that T is semi-simple if and only if the following is true:
If f is a polynomial and $f(T)$ is nilpotent, then $f(T) = 0$.
[5 marks]

4. Let θ be a real number. Prove that the following two matrices are similar over the field of complex numbers:
$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}.$$
[5 marks]

5. Show that the trace functional on $n \times n$ matrices is unique in the following sense. If W is the space of $n \times n$ matrices over the field F and if f is a linear functional on W such that

$f(AB) = f(BA)$ for each A and B in W, then f is a scalar multiple of the trace function. If, in addition, $f(I) = n$, then f is the trace function. [5 marks]

6. If T is a diagonalizable linear operator, then every T-invariant subspace has a complementary T-invariant subspace. [5 marks]

7. Let T be a linear operator on the finite-dimensional space V. Prove that T has a cyclic vector if and only if the following is true: Every linear operator U which commutes with T is a polynomial in T. [5 marks]